Wave turbulent statistics in non-weak wave turbulence

Yokoyama, Naoto

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Naoto Yokoyama*

Department of Aeronautics and Astronautics, Kyoto University, Kyoto 606-8501 JAPAN

Abstract

In wave turbulence, which is made by nonlinear interactions among waves, it has been believed that statistical properties are well described by the weak turbulence theory, where separation of linear and nonlinear time scales derived from weak nonlinearity is assumed. However, the separation of the time scales is often violated. To get rid of this inconsistency, closed equations are derived in wave turbulence without assuming the weak nonlinearity according to Direct-Interaction Approximation (DIA), which has been successful in Navier–Stokes turbulence. The DIA equations is a natural extension of the conventional kinetic equation to *not-necessarily-weak* wave turbulence.

Keywords: wave turbulence, turbulent statistics, direct-interaction approximation

1. Introduction

Energy in wave fields such as ocean surface waves [1], ocean internal waves [2], and elastic waves on thin metal plates [3] is transferred among wavenumbers owing to nonlinear interactions. The large degree-of-freedom wave fields are called wave turbulence. The weak turbulence theory, in which the nonlinear interactions are assumed to be small, is partially successful in wave turbulent statistics. Therefore, "the *wave* turbulence" and "the *weak* turbulence" are often regarded as synonyms.

However, the separation of the linear and nonlinear time scales, which is assumed in the weak turbulence theory, is often violated as pointed out based on the weak turbulence theory itself [4, 5, 6]. Especially in anisotropic wave turbulence such as ocean internal waves [2] and Alfvén waves [7], the separation of the time scales is almost always violated. The violations are observed also in both observations and direct numerical simulations in ocean surface waves [8] and ocean internal waves [9]. In the direct numerical simulations, the violations appear to be caused by fast non-resonant interactions.

*Tel: +81-774-38-3961, Fax: +81-774-38-3962
Email address: `yokoyama@kuaero.kyoto-u.ac.jp` (Naoto Yokoyama)

The weak and strong turbulence can coexist in anisotropic turbulent systems [10, 11]. The coexistence, which is called critical balance, is believed to emerge as a result of the scale-by-scale balance between the linear and nonlinear time scales. The coexistence is considered to be also in isotropic systems; the energy spectrum due to the coexistence in water gravity waves, for example, is known as Phillips' spectrum [12]. The critical balances in many systems is summarized in Ref. [13].

Therefore, the self-consistent statistical theory applicable to the non-weak wave turbulence, where the time separation is not satisfied, is required. We follow the Navier–Stokes turbulence statistics. One of the approach to this in Navier–Stokes turbulence is the eddy-damped quasi-normal Markovian (EDQNM) approximation [14, 15, 16]. Although the EDQNM is applicable to strongly nonlinear turbulence, it requires a semi-empirical parameter, which is eddy viscosity. Another is a self-contained statistical theory: the direct-interaction approximation (DIA) started by Kraichnan [17]. The original Eulerian DIA predicts a self-similar spectrum different from Kolmogorov spectrum [18]. The discrepancy results from the sweeping effect of large eddies on small eddies since the original Eulerian DIA equations violate the Galilean invariance and overestimate the nonlocal interactions between the small wavenumbers and the large wavenumbers. The Lagrangian DIA, where the sweeping is properly assessed, successfully predicts Kolmogorov spectrum in Navier–Stokes turbulence [19].

The sweeping is not always artificial since the nonlocal interactions play an essential role in energy transfer in several wave turbulence systems [20, 21]. Hence, in this Letter, in order to construct a statistical wave turbulence theory that accepts short-time and strong nonlinear interactions, DIA in Eulerian manner is applied to a general three-wave turbulent system. According to the DIA, closed equations for statistical wave turbulence without any empirical parameters are derived. The applicability of the DIA is also discussed.

2. Conventional Weak Turbulence Theory

The conventional weak turbulence theory is reviewed in this section. Wave turbulent systems with nonlinear interactions among three wavenumbers is generally governed by a canonical equation

$$\mathrm{i}\frac{\partial a(\boldsymbol{p})}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{p})} \tag{1}$$

for a complex amplitude $a(\boldsymbol{p})$ with Hamiltonian

$$\begin{aligned}\mathcal{H} = {} & (2\pi)^{-d}\int \mathrm{d}\boldsymbol{p}\ \omega|a(\boldsymbol{p})|^2 \\ & + (2\pi)^{-2d}\int \mathrm{d}\boldsymbol{p}\ \mathrm{d}\boldsymbol{p}_1\mathrm{d}\boldsymbol{p}_2 \left(\mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2} a^*(\boldsymbol{p})a(\boldsymbol{p}_1)a(\boldsymbol{p}_2) + \mathrm{c.c.}\right)\delta(\boldsymbol{p}-\boldsymbol{p}_1-\boldsymbol{p}_2),\end{aligned}$$

where $\boldsymbol{p}$ is a d-dimensional wavenumber vector, and $\omega(\boldsymbol{p})$ is a frequency of wavenumber $\boldsymbol{p}$ given by a linear dispersion relation. A matrix element $\mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}$

gives strength of three-wave nonlinear interactions among $\boldsymbol{p}$, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. A functional derivative with respect to a is expressed by $\delta/\delta a$. Moreover, a^* denotes the complex conjugate of a, and c.c. also denotes the complex conjugate of the previous term. Dirac's delta is denoted by $\delta(\boldsymbol{p})$. Since the wave media are assumed to have spacial inversion symmetry, $\omega(\boldsymbol{p}) = \omega(-\boldsymbol{p})$ and $\mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2} = \mathcal{T}^{-\boldsymbol{p}}_{-\boldsymbol{p}_1-\boldsymbol{p}_2}$.

A d-dimensional periodic wave field in a volume $(2\pi)^d$ is considered for simplicity. The canonical equation (1) is rewritten as

$$\begin{aligned}\frac{\partial a(\boldsymbol{p})}{\partial t} &= -\mathrm{i}\omega(\boldsymbol{p})a(\boldsymbol{p}) \\ &- \mathrm{i}\sum_{\boldsymbol{p}=\boldsymbol{p}_1+\boldsymbol{p}_2} \mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2} a(\boldsymbol{p}_1)a(\boldsymbol{p}_2) - 2\mathrm{i}\sum_{\boldsymbol{p}=-\boldsymbol{p}_1+\boldsymbol{p}_2} \mathcal{T}^{\boldsymbol{p}_2}_{\boldsymbol{p}\boldsymbol{p}_1}{}^* a^*(\boldsymbol{p}_1)a(\boldsymbol{p}_2). \end{aligned} \tag{2}$$

If the nonlinear terms in Eq. (2) is neglected, it can easily be integrated as

$$a(\boldsymbol{p}) = A(\boldsymbol{p})\exp(-\mathrm{i}\omega(\boldsymbol{p})t + \mathrm{i}\theta(\boldsymbol{p})).$$

Therefore, $a(\boldsymbol{p})$ is called complex amplitude and represents the behaviour of a mode $\boldsymbol{p}$ in the phase space.

The turbulent statistics governed by Eq. (2) is conventionally given according to the weak turbulence theory [22, 23]. Equation (2) is multiplied by $a^*(\boldsymbol{p}_4)$ and added complex conjugate with $\boldsymbol{p}$ and $\boldsymbol{p}_4$ interchanged, and the ensemble averaging $(\overline{\cdots})$ is taken:

$$\begin{aligned}\frac{\partial n(\boldsymbol{p})}{\partial t}\delta_{\boldsymbol{p}\boldsymbol{p}_4} = &- \mathrm{i}\sum_{\boldsymbol{p}=\boldsymbol{p}_1+\boldsymbol{p}_2} \mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2} \overline{a^*(\boldsymbol{p}_4)a(\boldsymbol{p}_1)a(\boldsymbol{p}_2)} \\ &- 2\mathrm{i}\sum_{\boldsymbol{p}=-\boldsymbol{p}_1+\boldsymbol{p}_2} \mathcal{T}^{\boldsymbol{p}_2}_{\boldsymbol{p}\boldsymbol{p}_1}{}^* \overline{a^*(\boldsymbol{p}_4)a^*(\boldsymbol{p}_1)a(\boldsymbol{p}_2)} \\ &+ \mathrm{c.c.}(\boldsymbol{p} \leftrightarrow \boldsymbol{p}_4), \end{aligned} \tag{3}$$

where $\mathrm{c.c.}(\boldsymbol{p} \leftrightarrow \boldsymbol{p}_4)$ denotes taking complex conjugates with $\boldsymbol{p}$ and $\boldsymbol{p}_4$ interchanged. Kronecker's delta is denoted as $\delta_{\boldsymbol{p}_i\boldsymbol{p}_j}$. The wave action $n(\boldsymbol{p})$, which is energy of wavenumber $\boldsymbol{p}$ divided by ω, is defined by $\overline{a(\boldsymbol{p}_i)a^*(\boldsymbol{p}_j)} = n(\boldsymbol{p}_i)\delta_{\boldsymbol{p}_i\boldsymbol{p}_j}$.

By applying the random phase approximation in the weak turbulence theory in the most primitive sense $\overline{a(\boldsymbol{p}_i)a^*(\boldsymbol{p}_j)a^*(\boldsymbol{p}_k)} = 0$ to the third-order correlation, the nonlinear term vanishes and

$$\frac{\partial n(\boldsymbol{p})}{\partial t} = 0.$$

Then, the non-zero third-order correlation is obtained by considering the time evolution of the third-order correlation. The time evolution of the third-order correlation is written as

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathrm{i}\Delta\omega^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}\right)\overline{aa_1^*a_2^*} &= -\mathrm{i}\left(\sum_{\boldsymbol{p}=\boldsymbol{p}_3+\boldsymbol{p}_4}\mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_3\boldsymbol{p}_4}\overline{a_1^*a_2^*a_3a_4}+2\sum_{\boldsymbol{p}+\boldsymbol{p}_3=\boldsymbol{p}_4}\mathcal{T}^{\boldsymbol{p}_4 *}_{\boldsymbol{p}\boldsymbol{p}_3}\overline{a_1^*a_2^*a_3^*a_4}\right)\\ &\qquad + \mathrm{c.c.}(\boldsymbol{p}\leftrightarrow\boldsymbol{p}_1)+\mathrm{c.c.}(\boldsymbol{p}\leftrightarrow\boldsymbol{p}_2)\\ &= -2\mathrm{i}\mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}\left(n_1n_2-n(n_1+n_2)\right). \end{aligned}\tag{4}$$

The random phase approximation is applied to the fourth-order correlation:

$$\begin{aligned}\overline{a_{\boldsymbol{p}_i}a_{\boldsymbol{p}_j}a^*_{\boldsymbol{p}_k}a^*_{\boldsymbol{p}_l}} &= \overline{a_{\boldsymbol{p}_i}a^*_{\boldsymbol{p}_k}}\,\overline{a_{\boldsymbol{p}_j}a^*_{\boldsymbol{p}_l}}+\overline{a_{\boldsymbol{p}_i}a^*_{\boldsymbol{p}_l}}\,\overline{a_{\boldsymbol{p}_j}a^*_{\boldsymbol{p}_k}}\\ &= n(\boldsymbol{p}_i)n(\boldsymbol{p}_j)\left(\delta_{\boldsymbol{p}_i\boldsymbol{p}_k}\delta_{\boldsymbol{p}_j\boldsymbol{p}_l}+\delta_{\boldsymbol{p}_i\boldsymbol{p}_l}\delta_{\boldsymbol{p}_j\boldsymbol{p}_k}\right).\end{aligned}$$

The frequency difference among the three wavenumbers is denoted by $\Delta\omega^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}=\omega(\boldsymbol{p})-\omega(\boldsymbol{p}_1)-\omega(\boldsymbol{p}_2)$.

The time variation of wave action n appearing in the right-hand side of Eq. (4) is possibly negligible by comparing with $1/\Delta\omega^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}$ if the nonlinearity is weak. Here, the linear timescales are assumed to be much faster than the nonlinear time scales. The separation of the time scales is nontrivial and is often violated.

When the separation of the time scales are valid, Eq. (4) can be integrated from t_0 to $t_0+\tau$, under the initial condition $\overline{aa_1^*a_2^*}(t_0)=0$. The third-order correlation is obtained as

$$\overline{aa_1^*a_2^*}=\frac{-2\mathrm{i}\mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}\left(n_1n_2-n(n_1+n_2)\right)\left(\exp(-\mathrm{i}\Delta\omega^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}\tau)-1\right)}{-\mathrm{i}\Delta\omega^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}}.$$

By employing

$$\frac{\mathrm{i}(\exp(-\mathrm{i}\Delta\omega\tau)-1)}{-\mathrm{i}\Delta\omega}=\mathrm{P.V.}\left(\frac{1}{\Delta\omega}\right)+\mathrm{i}\pi\delta(\Delta\omega)\qquad\text{as }\tau\to\infty,$$

Eq. (4) finally results in the kinetic equation:

$$\begin{aligned}\frac{\partial n}{\partial t} = 4\pi\Bigg(&\sum_{\boldsymbol{p}=\boldsymbol{p}_1+\boldsymbol{p}_2}|\mathcal{T}^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2}|^2\left(n_1n_2-n(n_1+n_2)\right)\delta(\Delta\omega^{\boldsymbol{p}}_{\boldsymbol{p}_1\boldsymbol{p}_2})\\ &-\sum_{\boldsymbol{p}_1=\boldsymbol{p}_2+\boldsymbol{p}}|\mathcal{T}^{\boldsymbol{p}_1}_{\boldsymbol{p}_2\boldsymbol{p}}|^2\left(n_2n-n_1(n_2+n)\right)\delta(\Delta\omega^{\boldsymbol{p}_1}_{\boldsymbol{p}_2\boldsymbol{p}})\\ &-\sum_{\boldsymbol{p}_2=\boldsymbol{p}+\boldsymbol{p}_1}|\mathcal{T}^{\boldsymbol{p}_2}_{\boldsymbol{p}\boldsymbol{p}_1}|^2\left(nn_1-n_2(n+n_1)\right)\delta(\Delta\omega^{\boldsymbol{p}_2}_{\boldsymbol{p}\boldsymbol{p}_1})\Bigg).\end{aligned}\tag{5}$$

The kinetic equation indicates that the energy is transferred among wavenumbers which satisfy the resonant conditions:

$$\begin{cases}\boldsymbol{p}=\boldsymbol{p}_1+\boldsymbol{p}_2\\ \omega=\omega_1+\omega_2\end{cases},\qquad \begin{cases}\boldsymbol{p}_1=\boldsymbol{p}_2+\boldsymbol{p}\\ \omega_1=\omega_2+\omega\end{cases},\qquad \begin{cases}\boldsymbol{p}_2=\boldsymbol{p}+\boldsymbol{p}_1\\ \omega_2=\omega+\omega_1\end{cases}.$$

3. Direct-Interaction Approximation for Wave Turbulence

As written in §2, the weak turbulence theory assumes, derived from the weak nonlinearity, that the time scales of the nonlinear energy transfer is much larger than the linear time scales. However, in almost all the anisotropic wave turbulence, the kinetic equation (5) itself gives short-time strong nonlinear interactions, and the separation of the linear and nonlinear time scales are often violated.

In this section, the direct-interaction approximation (DIA) [24], in which the separation of the time scales is not assumed but the largeness of the degrees of freedom is assumed, is applied to the wave turbulence. Note that the largeness of the degrees of freedom is also assumed implicitly in the weak turbulence theory. Instead of complex amplitude a, variables b_1 and b_2 defined by

$$b_1(\boldsymbol{p}) = \frac{a(\boldsymbol{p}) + a^*(-\boldsymbol{p})}{2}, \quad b_2(\boldsymbol{p}) = \frac{a(\boldsymbol{p}) - a^*(-\boldsymbol{p})}{2\mathrm{i}},$$

are employed. For example, the variables are defined as

$$b_1(\boldsymbol{p}) = \sqrt{\frac{\sigma|\boldsymbol{p}|}{2\rho}}\eta(\boldsymbol{p}), \quad b_2(\boldsymbol{p}) = \sqrt{\frac{\rho}{2\sigma|\boldsymbol{p}|}}\phi(\boldsymbol{p}),$$

where $\eta(\boldsymbol{p})$ and $\phi(\boldsymbol{p})$ are the Fourier transform of the surface elevation and that of the velocity potential in deep capillary waves [22], and they are

$$b_1(\boldsymbol{p}) = \frac{\sqrt{\omega}N_0}{\sqrt{2g|\boldsymbol{k}|}}\Pi(\boldsymbol{p}), \quad b_2(\boldsymbol{p}) = -\frac{\sqrt{g}|\boldsymbol{k}|}{\sqrt{2\omega}N_0}\phi(\boldsymbol{p}),$$

where $\Pi(\boldsymbol{p})$ and $\phi(\boldsymbol{p})$ are the Fourier transform of the stratification thickness and that of the velocity potential in ocean internal waves [25]. Since $b_i(\boldsymbol{p})$ is the Fourier transform of real functions, $b_i(\boldsymbol{p}) = b_i^*(-\boldsymbol{p})$.

The governing equation (2) is rewritten as

$$\frac{\partial b_i(\boldsymbol{p})}{\partial t} = \mathcal{L}_{ij}(\boldsymbol{p})b_j(\boldsymbol{p}) + \sum_{\boldsymbol{p}+\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{0}} \mathcal{N}_{ijk}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2) b_j(-\boldsymbol{p}_1) b_k(-\boldsymbol{p}_2). \tag{6}$$

The Einstein summation convention is employed. The linear and nonlinear matrix elements are

$$\begin{aligned} \mathcal{L}_{ij}(\boldsymbol{p}) &= (j-i)\omega(\boldsymbol{p}), \\ \mathcal{N}_{ijk}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2) &= \frac{\mathrm{i}^{i+j+k}}{2}\left((-1)^{i+1}\mathcal{T}^{\boldsymbol{p}}_{-\boldsymbol{p}_1-\boldsymbol{p}_2} + (-1)^j \mathcal{T}^{\boldsymbol{p}_1}_{-\boldsymbol{p}-\boldsymbol{p}_2} + (-1)^k \mathcal{T}^{\boldsymbol{p}_2}_{-\boldsymbol{p}-\boldsymbol{p}_1}\right) \\ &\quad + \left(\mathrm{c.c.}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2) \to (-\boldsymbol{p},-\boldsymbol{p}_1,-\boldsymbol{p}_2)\right), \end{aligned}$$

where $\mathrm{c.c.}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2) \to (-\boldsymbol{p},-\boldsymbol{p}_1,-\boldsymbol{p}_2)$ denotes taking complex conjugates and replacing $(\boldsymbol{p},\ \boldsymbol{p}_1,\ \boldsymbol{p}_2)$ by $(-\boldsymbol{p},\ -\boldsymbol{p}_1,\ -\boldsymbol{p}_2)$ simultaneously.

When a perturbation is added to the jth component of a wavenumber $\boldsymbol{p}'$ at a time t', which is $b_j(\boldsymbol{p}',t')$, the ith component of another wavenumber $\boldsymbol{p}$

at a later time t, which is $b_i(\boldsymbol{p},t)$, responds to the perturbation. The response function is defined as

$$G_{ij}(\boldsymbol{p},t|\boldsymbol{p}',t') = \frac{\delta b_i(\boldsymbol{p},t)}{\delta b_j(\boldsymbol{p}',t')}.$$

From Eq. (6), the governing equation of the response function is given by

$$\frac{\partial G_{in}(\boldsymbol{p},t|\boldsymbol{p}',t')}{\partial t} = \mathcal{L}_{ij}(\boldsymbol{p})G_{jn}(\boldsymbol{p},t|\boldsymbol{p}',t') + 2\sum_{\boldsymbol{p}+\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{0}} \mathcal{N}_{ijk}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2)b_j(-\boldsymbol{p}_1)G_{kn}(-\boldsymbol{p}_2,t|\boldsymbol{p}',t').$$

The initial condition of the response function is given as

$$G_{ij}(\boldsymbol{p},t'|\boldsymbol{p}',t') = \delta_{ij}\delta_{\boldsymbol{p}\boldsymbol{p}'}.$$

A perturbation that removes a triad interaction among $\boldsymbol{p}_0$, $\boldsymbol{q}_0$ and $\boldsymbol{r}_0$ that satisfies $\boldsymbol{p}_0+\boldsymbol{q}_0+\boldsymbol{r}_0=\boldsymbol{0}$ is added to a wave field at a time t'. When the degrees of freedom is large enough, the effect of the perturbation is small. The variable b_i is resolved into no direct-interaction field (NDI) where the direct interaction among $\boldsymbol{p}_0$, $\boldsymbol{q}_0$ and $\boldsymbol{r}_0$ is removed and direct-interaction field (DI). Then, $b_i = b_i^{(0)} + b_i^{(1)}$ and $|b_i^{(0)}| \gg |b_i^{(1)}|$. Similarly, the response function is also resolved into $G_{ij}^{(0)}$ in NDI and $G_{ij}^{(1)}$ in DI: $G_{ij} = G_{ij}^{(0)} + G_{ij}^{(1)}$ and $|G_{ij}^{(0)}| \gg |G_{ij}^{(1)}|$.

The equation of $b^{(0)}$ is given as

$$\frac{\partial b_i^{(0)}(\boldsymbol{p})}{\partial t} = \mathcal{L}_{ij}(\boldsymbol{p})b_j^{(0)}(\boldsymbol{p}) + \sum_{\substack{\boldsymbol{p}+\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{0}\\ \{\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2\}\neq\{\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0\}}} \mathcal{N}_{ijk}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2)b_j^{(0)}(-\boldsymbol{p}_1)b_k^{(0)}(-\boldsymbol{p}_2),$$

and that of $b^{(1)}$ is

$$\begin{aligned}
\frac{\partial b_i^{(1)}(\boldsymbol{p})}{\partial t} &= \mathcal{L}_{ij}(\boldsymbol{p})b_j^{(1)}(\boldsymbol{p}) \\
&+ 2\sum_{\substack{\boldsymbol{p}+\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{0}\\ \{\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2\}\neq\{\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0\}}} \mathcal{N}_{ijk}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2)b_j^{(0)}(-\boldsymbol{p}_1)b_k^{(1)}(-\boldsymbol{p}_2) \\
&+ \mathcal{N}_{ijk}(\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0)\left(\delta_{\boldsymbol{p}\boldsymbol{p}_0}b_j^{(0)}(-\boldsymbol{q}_0)b_k^{(0)}(-\boldsymbol{r}_0) + \delta_{\boldsymbol{p}\boldsymbol{p}_0}b_j^{(0)}(\boldsymbol{q}_0)b_k^{(0)}(\boldsymbol{r}_0)\right) \\
&+ \{\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0\}.
\end{aligned}$$

Here, $\{\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0\}$ denotes cyclic permutations from $(\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0)$. The governing equation for $G^{(0)}$ and $G^{(1)}$ is also obtained. The solution of $b^{(1)}$ is analytically obtained

$$\begin{aligned}
b_i^{(1)}(\boldsymbol{p}) &= \int_{t_0}^{t} \mathrm{d}t' \sum_{\boldsymbol{p}'} G_{in}^{(0)}(\boldsymbol{p},t|\boldsymbol{p}',t')\mathcal{N}_{njk}(\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0) \\
&\times \left(\delta_{\boldsymbol{p}'\boldsymbol{p}_0}b_j^{(0)}(-\boldsymbol{q}_0,t')b_k^{(0)}(-\boldsymbol{r}_0,t') + \delta_{\boldsymbol{p}'\boldsymbol{p}_0}b_j^{(0)}(\boldsymbol{q}_0,t')b_k^{(0)}(\boldsymbol{r}_0,t')\right) \\
&\qquad + \{\boldsymbol{p}_0,\boldsymbol{q}_0,\boldsymbol{r}_0\}
\end{aligned} \tag{7}$$

under the initial condition $b_i^{(1)}(t=t')=0$. The solution of $G^{(1)}$ is also obtained analytically and given by $b^{(0)}$ and $G^{(0)}$.

To investigate wave turbulent statistics, the correlation between b's defined as

$$\overline{V_{ij}(\boldsymbol{p},t,t')}=\overline{b_i(\boldsymbol{p},t)b_j^*(\boldsymbol{p},t')}$$

is considered. Because of Eq. (6) the simultaneous correlation is governed by the following equation:

$$\frac{\partial\overline{V_{ij}(\boldsymbol{p},t,t)}}{\partial t}=\mathcal{L}_{im}(\boldsymbol{p})\overline{V_{mj}(\boldsymbol{p},t,t)}+\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}}\mathcal{N}_{imn}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\overline{b_j(-\boldsymbol{p})b_m(-\boldsymbol{q})b_n(-\boldsymbol{r})}$$
$$+\,\text{c.c.}(i\leftrightarrow j). \tag{8}$$

The summation of m and n is taken over 1 and 2. DIA that removes the direct interactions among $\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}$ is applied to the wave field. First, Eq. (8) is rewritten as

$$\frac{\partial\overline{V_{ij}(\boldsymbol{p},t,t)}}{\partial t}=\mathcal{L}_{im}(\boldsymbol{p})\overline{V_{mj}(\boldsymbol{p},t,t)}$$
$$+\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}}\mathcal{N}_{imn}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\Big(\overline{b_j^{(0)}(-\boldsymbol{p})b_m^{(0)}(-\boldsymbol{q})b_n^{(0)}(-\boldsymbol{r})}+\overline{b_j^{(1)}(-\boldsymbol{p})b_m^{(0)}(-\boldsymbol{q})b_n^{(0)}(-\boldsymbol{r})}$$
$$+\overline{b_j^{(0)}(-\boldsymbol{p})b_m^{(1)}(-\boldsymbol{q})b_n^{(0)}(-\boldsymbol{r})}+\overline{b_j^{(0)}(-\boldsymbol{p})b_m^{(0)}(-\boldsymbol{q})b_n^{(1)}(-\boldsymbol{r})}\Big)$$
$$+\,\text{c.c.}(i\leftrightarrow j), \tag{9}$$

by resolving b in the right-hand side of Eq. (8) into $b^{(0)}$ (NDI) and $b^{(1)}$ (DI). In NDI field which has no interactions among $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$, $\overline{b_j^{(0)}(-\boldsymbol{p})b_m^{(0)}(-\boldsymbol{q})b_n^{(0)}(-\boldsymbol{r})}=0$, since $b_j^{(0)}(\boldsymbol{p})$, $b_m^{(0)}(\boldsymbol{q})$, $b_n^{(0)}(\boldsymbol{r})$ are statistically independent. Second, the solutions (7) are substituted into $\overline{b_j^{(1)}(-\boldsymbol{p})b_m^{(0)}(-\boldsymbol{q})b_n^{(0)}(-\boldsymbol{r})}$ and similar terms. At last, Eq. (8) is expressed by the different-time correlation and the response function:

$$\frac{\partial\overline{V_{ij}(\boldsymbol{p},t,t)}}{\partial t}=\mathcal{L}_{im}(\boldsymbol{p})\overline{V_{mj}(\boldsymbol{p},t,t)}$$
$$+\,2\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}}\mathcal{N}_{imn}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\int_{t_0}^{t}\mathrm{d}t'\Big(\mathcal{N}_{abc}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\overline{G_{ja}(-\boldsymbol{p},t|-\boldsymbol{p},t')}\;\overline{V_{bm}(\boldsymbol{q},t',t)}\;\overline{V_{cn}(\boldsymbol{r},t',t)}$$
$$+\,\mathcal{N}_{abc}(\boldsymbol{q},\boldsymbol{r},\boldsymbol{p})\overline{G_{ma}(-\boldsymbol{q},t|-\boldsymbol{q},t')}\;\overline{V_{bn}(\boldsymbol{r},t',t)}\;\overline{V_{cj}(\boldsymbol{p},t',t)}$$
$$+\mathcal{N}_{abc}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{q})\overline{G_{na}(-\boldsymbol{r},t|-\boldsymbol{r},t')}\;\overline{V_{bj}(\boldsymbol{p},t',t)}\;\overline{V_{cm}(\boldsymbol{q},t',t)}\Big)$$
$$+\,\text{c.c.}(i\leftrightarrow j). \tag{10}$$

The statistical independence between b_i and G_{mn} is assumed.

A similar procedure can be applied to the different-time correlation. Hence, the governing equation of the different-time correlation is obtained as

$$
\begin{aligned}
&\frac{\partial\overline{V_{ij}(\boldsymbol{p},t,t')}}{\partial t} = \mathcal{L}_{im}(\boldsymbol{p})\overline{V_{mj}(\boldsymbol{p},t,t')} \\
&+ 2\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}}\mathcal{N}_{imn}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\left(\int_{t_0}^{t'}\mathrm{d}t''\mathcal{N}_{abc}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\overline{G_{ja}(-\boldsymbol{p},t'|-\boldsymbol{p},t'')}\ \overline{V_{bm}(\boldsymbol{q},t'',t)}\ \overline{V_{cn}(\boldsymbol{r},t'',t)}\right. \\
&+\int_{t_0}^{t}\mathrm{d}t''\left(\mathcal{N}_{abc}(\boldsymbol{q},\boldsymbol{r},\boldsymbol{p})\overline{G_{ma}(-\boldsymbol{q},t|-\boldsymbol{q},t'')}\ \overline{V_{bn}(\boldsymbol{r},t'',t)}\ \overline{V_{cj}(\boldsymbol{p},t'',t')}\right. \\
&\left.\left.+\mathcal{N}_{abc}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{q})\overline{G_{na}(-\boldsymbol{r},t|-\boldsymbol{r},t'')}\ \overline{V_{bj}(\boldsymbol{p},t'',t')}\ \overline{V_{cm}(\boldsymbol{q},t'',t)}\right)\right). \qquad (11)
\end{aligned}
$$

Moreover, the governing equation of the response function is also obtained as

$$
\begin{aligned}
&\frac{\partial\overline{G_{in}(\boldsymbol{p},t|\boldsymbol{p},t')}}{\partial t} = \mathcal{L}_{ij}(\boldsymbol{p})\overline{G_{jn}(\boldsymbol{p},t|\boldsymbol{p},t')} \\
&+ 4\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}}\mathcal{N}_{ijk}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\mathcal{N}_{abc}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{q})\int_{t'}^{t}\mathrm{d}t''\overline{G_{ka}(-\boldsymbol{r},t|-\boldsymbol{r},t'')}\ \overline{G_{bn}(\boldsymbol{p},t''|\boldsymbol{p},t')}\ \overline{V_{cj}(\boldsymbol{q},t'',t)}. \qquad (12)
\end{aligned}
$$

Equations (10–12) give a closed equation system and they are the DIA equations of the wave turbulence.

It should be emphasized that the weak nonlinearity is not assumed in the procedure. Therefore, the DIA equations of the wave turbulence (10–12) can be applied also to strongly nonlinear wave turbulent systems.

4. DIA Equations for Autocorrelation of Complex Amplitude

In the previous section, DIA is applied to the variable b. Turbulent statistics is described with the complex amplitude a to compare the weak turbulence theory in this section. The correlation of a is defined as $\overline{a(\boldsymbol{p},t)a^*(\boldsymbol{p},t')} = N(\boldsymbol{p},t,t')$ and $\overline{a(\boldsymbol{p},t)a(-\boldsymbol{p},t')} = M(\boldsymbol{p},t,t')$. The correlation of b, that is, $\overline{V_{ij}}$ is expressed by M and N as

$$
\begin{aligned}
\overline{V_{ij}(\boldsymbol{p},t,t')} = \frac{1}{4}\Big(&\mathrm{i}^{-(i-j)}N(\boldsymbol{p},t,t') + \mathrm{i}^{i-j}N^*(-\boldsymbol{p},t,t') \\
&-\left(\mathrm{i}^{-(i+j)}M(\boldsymbol{p},t,t') + \mathrm{i}^{i+j}M^*(-\boldsymbol{p},t,t')\right)\Big).
\end{aligned}
$$

The initial condition of the cross-correlation is that $M(\boldsymbol{p},t_0,t_0) = 0$ since $a(\boldsymbol{p})$ and $a(-\boldsymbol{p})$ are uncorrelated initially. The cross-correlation at later time is much smaller than the auto-correlation, $|M(\boldsymbol{p},t,t')| \ll |N(\boldsymbol{p},t,t')|$. Therefore,

$$
\begin{aligned}
\overline{V_{ij}(\boldsymbol{p},t|\boldsymbol{p},t')} &= \frac{1}{4}\left(\mathrm{i}^{-(i-j)}N(\boldsymbol{p},t,t') + \mathrm{i}^{i-j}N^*(-\boldsymbol{p},t,t')\right), \\
\overline{G_{ij}(\boldsymbol{p},t|\boldsymbol{p},t')} &= \frac{1}{2}\left(\mathrm{i}^{-(i-j)}G(\boldsymbol{p},t,t') + \mathrm{i}^{i-j}G^*(-\boldsymbol{p},t,t')\right).
\end{aligned}
$$

Equation (10) is rewritten to a equation for $N(\boldsymbol{p},t,t)$ as

$$\begin{aligned}\frac{\partial N(\boldsymbol{p},t,t)}{\partial t} =& 2\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}} \int_{t_0}^{t} \mathrm{d}t' \, (G^*(\boldsymbol{p},t,t') \, \left(|\mathcal{T}^{\boldsymbol{p}}_{-\boldsymbol{q}-\boldsymbol{r}}|^2 N^*(-\boldsymbol{q},t',t)N^*(-\boldsymbol{r},t',t)\right. \\ &+|\mathcal{T}^{-\boldsymbol{q}}_{\boldsymbol{r}\boldsymbol{p}}|^2 N^*(-\boldsymbol{q},t',t)N(\boldsymbol{r},t',t) + |\mathcal{T}^{-\boldsymbol{r}}_{\boldsymbol{p}\boldsymbol{q}}|^2 N(\boldsymbol{q},t',t)N^*(-\boldsymbol{r},t',t)\big) \\ &- N(\boldsymbol{p},t',t)\left(|\mathcal{T}^{\boldsymbol{p}}_{-\boldsymbol{q}-\boldsymbol{r}}|^2 G(-\boldsymbol{q},t,t')N^*(-\boldsymbol{r},t',t)\right. \\ &+|\mathcal{T}^{-\boldsymbol{q}}_{\boldsymbol{r}\boldsymbol{p}}|^2 G(-\boldsymbol{q},t,t')N(\boldsymbol{r},t',t) - |\mathcal{T}^{-\boldsymbol{r}}_{\boldsymbol{p}\boldsymbol{q}}|^2 G^*(\boldsymbol{q},t,t')N^*(-\boldsymbol{r},t',t)\big) \\ &+\{\boldsymbol{q}\leftrightarrow\boldsymbol{r}\}) + \text{c.c.} \end{aligned} \tag{13}$$

Similarly, the different-time correlation $N(\boldsymbol{p},t,t')$ is also obtained from Eq. (11). Moreover, Eq. (12) can be expressed by N and G. Namely, the DIA equations (10–12) in the wave turbulence as equations for the correlation $\overline{V_{ij}}$ of b can be rewritten as equations for the correlation N for a.

Furthermore, the fluctuation–dissipation relation, $N(\boldsymbol{p},t,t') = n(\boldsymbol{p},t')G(\boldsymbol{p},t,t')$, is employed. The simultaneous correlation is written as $N(\boldsymbol{p},t,t) = n(\boldsymbol{p},t)$ from now on. Hence, the DIA equations (10–12) can be rewritten as

$$\begin{aligned}\frac{\partial n(\boldsymbol{p},t)}{\partial t} =& 2\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}} \int_{t_0}^{t} \mathrm{d}t' \, \left(|\mathcal{T}^{\boldsymbol{p}}_{-\boldsymbol{q}-\boldsymbol{r}}|^2 \left(n(-\boldsymbol{q},t')n(-\boldsymbol{r},t') - n(\boldsymbol{p},t')\left(n(-\boldsymbol{q},t')+n(-\boldsymbol{r},t')\right)\right)\right. \\ & G^*(\boldsymbol{p},t,t')G(-\boldsymbol{q},t,t')G(-\boldsymbol{r},t,t') \\ &-|\mathcal{T}^{-\boldsymbol{q}}_{\boldsymbol{r}\boldsymbol{p}}|^2 \left(n(\boldsymbol{p},t')n(\boldsymbol{r},t') - n(-\boldsymbol{q},t')\left(n(\boldsymbol{p},t')+n(\boldsymbol{r},t')\right)\right) \, G^*(\boldsymbol{p},t,t')G(-\boldsymbol{q},t,t')G^*(\boldsymbol{r},t,t') \\ &-|\mathcal{T}^{-\boldsymbol{r}}_{\boldsymbol{p}\boldsymbol{q}}|^2 \left(n(\boldsymbol{p},t')n(\boldsymbol{q},t') - n(-\boldsymbol{r},t')\left(n(\boldsymbol{p},t')+n(\boldsymbol{q},t')\right)\right) \, G^*(\boldsymbol{p},t,t')G^*(\boldsymbol{q},t,t')G(-\boldsymbol{r},t,t')\big) \\ &+ \text{c.c.} \end{aligned} \tag{14}$$

and

$$\begin{aligned}\frac{\partial G(\boldsymbol{p},t,t')}{\partial t} =& -\mathrm{i}\omega(\boldsymbol{p})G(\boldsymbol{p},t,t') \\ &- 2\sum_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0}} \int_{t'}^{t} \mathrm{d}t'' G(\boldsymbol{p},t'',t') \left(|\mathcal{T}^{\boldsymbol{p}}_{-\boldsymbol{q}-\boldsymbol{r}}|^2 \left(n(-\boldsymbol{q},t'')+n(-\boldsymbol{r},t'')\right) G(-\boldsymbol{q},t,t'')G(-\boldsymbol{r},t,t'')\right. \\ &\qquad - |\mathcal{T}^{-\boldsymbol{q}}_{\boldsymbol{r}\boldsymbol{p}}|^2 \left(n(-\boldsymbol{q},t'')-n(\boldsymbol{r},t'')\right) G(-\boldsymbol{q},t,t'')G^*(\boldsymbol{r},t,t'') \\ &\qquad \left.-|\mathcal{T}^{-\boldsymbol{r}}_{\boldsymbol{p}\boldsymbol{q}}|^2 \left(n(-\boldsymbol{r},t'')-n(\boldsymbol{q},t'')\right) G^*(\boldsymbol{q},t,t'')G(-\boldsymbol{r},t,t'')\right). \end{aligned} \tag{15}$$

Here, $t_0 = t'$ thanks to the causality. Equations (14) and (15) are another set of the DIA equations.

Equation (14) in the DIA equations and the conventional kinetic equation look quite similar. Nonetheless, there exist some differences. Hysteresis is included in the DIA equations since the time integration is incorporated in Eq. (14). It is in contrast with the Markovian properties of the kinetic equation. Furthermore, the response functions appear in the integrand of Eq. (14). Hence, the DIA equations accept nonlinear changes of phases. The intermittent structures such as freak waves which develop abruptly have the non-Markovian properties and the phase entrainment. Therefore, it is expected that the intermittency is evaluated statistically by the DIA equations.

The quadratic energy $\sum_{\boldsymbol{p}} \omega(\boldsymbol{p}) n(\boldsymbol{p})$ is not strictly conserved for Eq. (14) since Hamiltonian which is the sum of quadratic and cubic energies is conserved. The quadratic energy is conserved for the kinetic equation (5) owing to the resonant interactions. Therefore, this non-conservation laws implies that the fast non-resonant interactions associated with strongly nonlinear coherent structures can be statistically estimated by the nonlinear parts of the response functions. The numerical simulations of Eqs. (14) and (15) are required to evaluate the fast non-resonant interactions for the specific wave turbulent systems. The quadratic momentum $\sum_{\boldsymbol{p}} \boldsymbol{p} n(\boldsymbol{p})$ is a conserved quantity because of the conditions of the wavenumbers. Therefore, one can find an equilibrium solution $n(\boldsymbol{p}) \propto (\boldsymbol{p} \cdot \boldsymbol{U})^{-1}$, where $\boldsymbol{U}$ is an arbitrary constant vector. The entropy $\sum_{\boldsymbol{p}} \log n(\boldsymbol{p})$ never decrease in Eq. (14). In this manner, some of statistical natures are the same as the conventional kinetic equation (5).

For the short-time limit $t \to t_0$, Eq. (14) is rewritten as

$$\begin{aligned}
\frac{\partial n(\boldsymbol{p})}{\partial t} \approx & 4(t-t_0)\left(\sum_{\boldsymbol{p}=\boldsymbol{q}+\boldsymbol{r}} \left|\mathcal{T}_{\boldsymbol{q}\boldsymbol{r}}^{\boldsymbol{p}}\right|^2 \left(n(\boldsymbol{q})n(\boldsymbol{r}) - n(\boldsymbol{p})(n(\boldsymbol{q})+n(\boldsymbol{r}))\right)\right. \\
& - \sum_{\boldsymbol{q}=\boldsymbol{r}+\boldsymbol{p}} \left|\mathcal{T}_{\boldsymbol{r}\boldsymbol{p}}^{\boldsymbol{q}}\right|^2 \left(n(\boldsymbol{r})n(\boldsymbol{p}) - n(\boldsymbol{q})(n(\boldsymbol{r})+n(\boldsymbol{p}))\right) \\
& \left. - \sum_{\boldsymbol{r}=\boldsymbol{p}+\boldsymbol{q}} \left|\mathcal{T}_{\boldsymbol{p}\boldsymbol{q}}^{\boldsymbol{r}}\right|^2 \left(n(\boldsymbol{p})n(\boldsymbol{q}) - n(\boldsymbol{r})(n(\boldsymbol{p})+n(\boldsymbol{q}))\right)\right). \qquad (16)
\end{aligned}$$

This is also consistent with the short-time kinetic equation in Ref. [23]. Equation (16) is valid even for strongly nonlinear regimes since the equation is derived without employing the separation of the linear and nonlinear time scales.

5. Recovery of Kinetic Equation from DIA Equations

To recover the kinetic equation from Eq. (13), the weak nonlinearity is assumed as an "extra" assumption in this section. The time variations of the different-time correlation and the response function are as follows:

$$\begin{aligned}
\frac{\partial N(\boldsymbol{p},t,t')}{\partial t} &= -\mathrm{i}\omega N(\boldsymbol{p},t,t') + \text{nonlinear terms}, \\
\frac{\partial G(\boldsymbol{p},t,t')}{\partial t} &= -\mathrm{i}\omega G(\boldsymbol{p},t,t') + \text{nonlinear terms}.
\end{aligned}$$

The nonlinear terms of the different-time correlation and the response function do not contribute to the simultaneous correlation at the leading order. Then, by neglecting the nonlinear terms, the leading terms of the different-time correlation and the response function are obtained as

$$N(\boldsymbol{p},t,t') = n(\boldsymbol{p},t')\mathrm{e}^{-\mathrm{i}\omega(t-t')}, \quad G(\boldsymbol{p},t,t') = \mathrm{e}^{-\mathrm{i}\omega(t-t')},$$

under the assumption of the weak nonlinearity. Therefore, since $n(\boldsymbol{p},t)$ varies much slower than $1/\Delta\omega$, $n(\boldsymbol{p},t) = n(\boldsymbol{p},t') = n(\boldsymbol{p})$. This $n(\boldsymbol{p})$ is the very wave action in the weak turbulence theory. By substituting the leading terms of the different-time correlation and the response function into Eq. (13), we obtain

$$\begin{aligned}\frac{\partial n(\boldsymbol{p})}{\partial t} =& 2\int_{t_0}^{t} \mathrm{d}t' \left(\sum_{\boldsymbol{p}=\boldsymbol{q}+\boldsymbol{r}} \left|\mathcal{T}_{\boldsymbol{qr}}^{\boldsymbol{p}}\right|^2 \left(n(\boldsymbol{q})n(\boldsymbol{r}) - n(\boldsymbol{p})(n(\boldsymbol{q})+n(\boldsymbol{r}))\right) \mathrm{e}^{\mathrm{i}\Delta\omega_{\boldsymbol{qr}}^{\boldsymbol{p}}(t-t')} \right.\\ &- \sum_{\boldsymbol{q}=\boldsymbol{r}+\boldsymbol{p}} \left|\mathcal{T}_{\boldsymbol{rp}}^{\boldsymbol{q}}\right|^2 \left(n(\boldsymbol{r})n(\boldsymbol{p}) - n(\boldsymbol{q})(n(\boldsymbol{r})+n(\boldsymbol{p}))\right) \mathrm{e}^{-\mathrm{i}\Delta\omega_{\boldsymbol{rp}}^{\boldsymbol{q}}(t-t')}\\ &\left.- \sum_{\boldsymbol{r}=\boldsymbol{p}+\boldsymbol{q}} \left|\mathcal{T}_{\boldsymbol{pq}}^{\boldsymbol{r}}\right|^2 \left(n(\boldsymbol{p})n(\boldsymbol{q}) - n(\boldsymbol{r})(n(\boldsymbol{p})+n(\boldsymbol{q}))\right) \mathrm{e}^{-\mathrm{i}\Delta\omega_{\boldsymbol{pq}}^{\boldsymbol{r}}(t-t')} \right)\\ &+ \mathrm{c.c.}\end{aligned} \tag{17}$$

Since the separation of the time scales is assumed, t_0 can be set to $-\infty$. Then,

$$\int_{t_0}^{t} \mathrm{d}t' \mathrm{e}^{\mathrm{i}\Delta\omega_{\boldsymbol{qr}}^{\boldsymbol{p}}(t-t')} = \mathrm{i}\left(\mathrm{P.V.}\left(\frac{1}{\Delta\omega_{\boldsymbol{qr}}^{\boldsymbol{p}}}\right) - \mathrm{i}\pi\delta(\Delta\omega_{\boldsymbol{qr}}^{\boldsymbol{p}})\right). \tag{18}$$

Finally, Eq. (17) results in Eq. (5). Namely, the kinetic equation in the weak turbulence theory can be recovered by an additional assumption, that is, the weak nonlinearity to the DIA equations.

6. Concluding Remark

6.1. Discussion

In the short-time limit, Eq. (16) is consistent with what is derived along the weak turbulence theory. The conventional kinetic equation is also recovered from the DIA equations in the weakly nonlinear limit. The nonlinear parts of the response function are irrelevant to the time variation of the wave action in both limits.

As pointed out in direct numerical simulations of four-wave weak turbulent system [26], the convergence to the kinetic equation is much faster than the convergence of the integral to the δ function in Eq. (18). During the intermediate time, which is longer than the short time and shorter than the convergence, the nonlinear parts of the response function will play a role. The nonlinear behaviours of the response function will be investigated for specific systems with numerical simulations.

The procedure made in §3 corresponds to the Eulerian DIA in Navier–Stokes turbulence. The Eulerian DIA in Navier–Stokes turbulence predicts an incorrect self-similar spectrum because of the violation of the Galilean invariance. Therefore, the DIA in Letter may not be applicable to turbulent statistics in the systems where coherent structures make broad-band spectra. Conversely, the DIA in wave turbulence is expected to be successful in statistical description of the wave turbulence such that the essential interactions are nonlocal in the wavenumber space, the time scale of the local interactions are short like freak waves, the inertial subrange is small, or the energy of large-scale waves are not greater than that of small-scale waves.

In wave turbulent systems where coherent structures make broad-band spectra, modification to the DIA like Lagrangian DIA will be successful as the Lagrangian DIA in Navier–Stokes turbulence overcomes the failure of the Eulerian DIA by introducing Lagrangian velocity correlation. If the critical balance is made by the local interactions, the modification is imperative. However, Lagrangian description is not available in non-hydrodynamic systems. It is also difficult to be employed in systems with spacial dimension different from dimension of wavenumbers such as surface waves. Then, appropriate gauges specific to the system must be introduced in these systems.

6.2. Conclusion

The closed equation system is developed for the not-necessarily-weak wave turbulence statistics according to direct-interaction approximation (DIA). In the procedure, the three assumptions below are made:

- Quantities in the field without the direct interactions are much larger than that in the perturbed field under the largeness of the degrees of freedom in the wave field.
- $b_j^{(0)}(\boldsymbol{p})$, $b_m^{(0)}(\boldsymbol{q})$ and $b_n^{(0)}(\boldsymbol{r})$ are statistically independent in the field without the direct interactions among $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$.
- b_i and G are statistically independent.

By developing the equation system without assuming the weak nonlinearity, the DIA equations can consistently make statistical description for the wave turbulent system that has even short-time and strong nonlinear interactions. The separation of the linear and nonlinear time scales is the key assumption made in the conventional weak turbulence theory. The DIA equations provide us an appropriate tool to evaluate spatio-temporal intermittent structures like freak waves in the wave turbulent system. The intermittency, which is considered to emerge when the linear and nonlinear time scales are balanced, is difficult to be consistently evaluated by the conventional weak turbulence theory. The DIA equations can also be applied to the wave turbulent statistics of the nonlocally-interacting systems, which may be established by mechanisms other than step-by-step cascades in Naver–Stokes turbulence.

The kinetic equation in the weak turbulence theory is also recovered from the DIA equations. This indicates that the framework of the DIA which harness the largeness of the degrees of freedom is the natural extension of the weak turbulence theory.

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References

[1] V. E. Zakharov, J. Appl. Mech. Tech. Phys. 2 (1968) 190–194.

[2] Y. Lvov, E. Tabak, K. Polzin, N. Yokoyama, J. Phys. Oceanogr. 40 (2010) 2605–2623.

[3] G. Düring, C. Josserand, S. Rica, Phys. Rev. Lett. 97 (2006) 25503.

[4] A. C. Newell, S. Nazarenko, L. Biven, Physica D 152–153 (2001) 520–550.

[5] C. Connaughton, S. Nazarenko, A. C. Newell, Physica D 184 (2003) 86–97.

[6] G. Düring, A. Picozzi, S. Rica, Physica D 238 (2009) 1524–1549.

[7] S. V. Nazarenko, A. C. Newell, S. Galtier, Physica D 152–153 (2001) 646–652.

[8] N. Yokoyama, J. Fluid Mech. 501 (2004) 169–178.

[9] Y. V. Lvov, N. Yokoyama, Physica D 238 (2009) 803–815.

[10] P. Goldreich, S. Sridhar, Astrophys. J. 438 (1995) 763–775.

[11] S. V. Nazarenko, A. A. Schekochihin, J. Fluid Mech. 677 (2011) 134–153.

[12] O. M. Phillips, J. Fluid Mech. 4 (1958) 426–434.

[13] S. Nazarenko, Wave Turbulence, Springer, Heidelberg, 2011.

[14] F. Waleffe, Phys. Fluids A 5 (1993) 677–685.

[15] C. Cambon, R. Rubinstein, F. S. Godeferd, New J. Phys. 6 (2004) 73.

[16] P. Sagaut, C. Cambon, Homogeneous turbulence dynamics, Cambridge University Press, Cambridge, 2008.

[17] R. H. Kraichnan, Phys. Rev. 109 (1958) 1407–1422.

[18] R. H. Kraichnan, J. Fluid Mech. 5 (1959) 497–543.

[19] S. Kida, S. Goto, J. Fluid Mech. 345 (1997) 307–345.

[20] P. S. Iroshnikov, Sov. Astron. 7 (1964) 566–571.

[21] C. David, A. J. Majda, D. W. McLaughlin, E. G. Tabak, Physica D 152–153 (2001) 551–572.

[22] V. E. Zakharov, V. S. L'vov, G. Falkovich, Kolmogorov Spectra of Turbulence I: Wave Turbulence, Springer-Verlag, Berlin, 1992.

[23] P. A. E. M. Janssen, J. Phys. Oceanogr. 33 (2003) 863–884.

[24] S. Goto, S. Kida, Physica D 117 (1998) 191–214.

[25] Y. V. Lvov, E. G. Tabak, Phys. Rev. Lett. 87 (2001) 168501.

[26] M. Tanaka, J. Phys. Oceanogr. 37 (2007) 1022–1036.